

A Study on the Sudden Death of Entanglement

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(Dated: February 1, 2008)

The dynamics of entanglement and the phenomenon of entanglement sudden death (ESD) [1] are discussed in bipartite systems, measured by Wootters Concurrence. Our calculation shows that ESD appears whenever the system is open or closed and is dependent on the initial condition. The relation of the evolution of entanglement and energy transfer between the system and its surroundings is also studied.

PACS numbers: 03.65.Yz, 03.65.Ud, 03.67.Mn

I. INTRODUCTION

The dynamics of entanglement in bipartite systems has received great attention since the work of Yu and Eberly [2], in which the entanglement between the two particles coupled with two independent environments became completely vanishing in a finite time. This surprising phenomenon, contrary to intuition based on experience about qubit decoherence, intrigues great interests [3, 4, 5, 6, 7, 8]. Different from the original work of Yu and Eberly [2], in which two independent particles embedded into its own dissipative environment and there is no any direct or indirect interaction, the effects of interaction between the particles and the couplings to the same environment have been discussed extensively in Ref. [4] and [5]. In [4], the authors show that for a special initial state the entanglement disappeared in a finite time and then revived after a dark period because of the interaction between the particles. Furthermore the authors show that the entanglement sudden death (ESD) is sensitive to the initial condition, as proved in [5]. An important character in [4, 5] is that the environment is dissipative and the transfer of energy between the system and environment is inevitable. Another important situation is the dephasing environment, in which energy transfer from the system to the environment does not occur. Some works have been devoted to this issue [3, 7]. In [3], the authors show that disentanglement is dependent on the initial condition and temperature of the environment.

Although the extensive progress in understanding the disentanglement, it is still unclear what reason causes this phenomenon and what is the physics behind. In this paper we try to give an enlightening discussion by examining some examples. In the previous studies, the couplings with the environment has been considered as the critical point of disentanglement and ESD. However, as will find in this paper, ESD can also happen in closed systems. In our own point, the phenomena of disentanglement and ESD stem from the dissipative terms in the Hamiltonian, independent of the coupling with environ-

ment, and are sensitive to the initial conditions. Moreover the revival of entanglement can also be explained as the effect of the backaction terms in the Hamiltonian.

The paper is organized as follows. In Sec. II, we first review the dynamics of entanglement in (a) Tavis-Cumming model [9] and (b) dephasing system respectively. In Sec. III, the dynamics of entanglement in closed bipartite system has been discussed, and ESD and revivals of entanglement have also been observed in this system. The further discussions and conclusions is presented in the final section.

II. DISENTANGLEMENT IN OPEN SYSTEMS

The dynamics of entanglement in open systems has been discussed extensively [1, 2, 4, 5, 6, 7]. However the physics behind the evolution of entanglement is rarely touched. In this section we try to present a physical interpretation for concurrence by examining two examples. The first is the Tavis-Cumming model [9], in which two non-interaction qubits couple with the same quantized field under the rotating-wave approximation. Another is the dephasing model, in which two qubits embed into a multimode quantized field and the interaction between the two qubits is also considered.

A. Tavis-Cumming Model

For two spin-1/2 particles coupled with a single-mode cavity field, the Hamiltonian is written as [9],

$$H = \frac{\omega_0}{2}(\sigma_A^z + \sigma_B^z) + \omega a^\dagger a + g \sum_{i=A,B} (a\sigma_i^+ + a^\dagger\sigma_i^-), \quad (1)$$

where $a^{(\dagger)}$ is bosonic operator and σ_i^\pm are the rising and lowering operator for spin-1/2. We focus our discussion on the resonating case ($\omega_0 = \omega$), in that the spectrum of Hamiltonian can be obtained exactly. For the whole system (system + field), the evolution of the whole system is characterized by the interaction between the system and field. However from the point of the system (two spin-1/2 particles), the energy transfer between the system and the field happens, which is described by the relaxation term

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$a^\dagger \sigma_i^-$ and the backaction term $a \sigma_i^+$. This example is very different from Refs [2, 7], in which the two qubits couples with two independent environments separately. In fact the couplings with the same field could induce an effective interaction between two qubits, which manipulates the entanglement between the two qubits [11]. Furthermore, as will be shown in the following, the phenomenon of ESD is sensitive to the initial conditions, and the energy transfer because of the interaction is directly related to the disentanglement and revival of the entanglement.

The evolution operator $U(t) = \exp(-iHt)$ can be calculated exactly [10]. Choose the initial state $\rho(0) = (\frac{1-r}{4} + r|\varphi\rangle\langle\varphi|) \otimes |0\rangle\langle 0|$, in which $|\varphi\rangle = \sin\theta|ee\rangle + \cos\theta|gg\rangle$ with $|e(g)\rangle$ denoting two eigenstates of spin-1/2 particles, and $|0\rangle$ is the vacuum state for the quantized field. The entanglement in the initial state is measured by the purity r and mixing θ . Then the density matrix for the system $\varrho(t) = \text{Tr}_E[U(t)\rho(0)U^\dagger(t)]$ can be expressed in the basis $\{|1\rangle = |ee\rangle, |2\rangle = |gg\rangle, |3\rangle = |eg\rangle, |4\rangle = |ge\rangle\}$,

$$\begin{aligned}\varrho_{11} &= \frac{1-r}{4} + r(\sin\theta \frac{\cos\sqrt{6}gt+2}{3})^2, \\ \varrho_{22} &= \frac{1-r}{4} + r\cos^2\theta + 2r(\sin\theta \frac{\cos\sqrt{6}gt-1}{3})^2, \\ \varrho_{12} &= \varrho_{21} = r \frac{\cos\sqrt{6}gt+2}{3} \sin\theta \cos\theta, \\ \varrho_{33} &= \varrho_{44} = \frac{1-r}{4} + r \frac{\sin^2\sqrt{6}gt \sin^2\theta}{6}, \\ \varrho_{34} &= \varrho_{43} = r \frac{\sin^2\sqrt{6}gt \sin^2\theta}{6}.\end{aligned}\quad (2)$$

The concurrence $c = \max\{0, \varrho_{34} - \sqrt{\varrho_{11}\varrho_{22}}\}$ has been calculated, as shown in Fig.1, for different initial states. Obviously the concurrence is fluctuating with the rescaled time gt and can be zero in a finite time. We have highlight the points with red color that the concurrence is vanishing, as the so-called entanglement sudden death (ESD), in Fig.1 in this plot. An important point is that ESD is sensitive to the initial state as displayed in Fig.1. For the mixed initial state $r < 1$, the ESD happens readily (Fig.1b). However, for the pure initial states, the region with vanishing concurrence is compressed greatly (Fig.1a). This phenomenon shows that ESD is completely initial-state sensitivity in this model.

Another important character revealed by Fig.1 is the fluctuation/revival of entanglement. This phenomenon have been discussed qualitatively in Ref. [4]. However a clearer interpretation is absent; How the entanglement is constructed by the interaction and why and where the entanglement decreases with the evolution? What is the physics behind the dynamics of entanglement? In this part we try to give an enlightening discussion for this problem based on this model. Let us first review the Hamiltonian Eq. (1); Obviously the dynamics of the two independent spin-1/2 particles is determined by the interaction terms in Eq. (1), which is the charge of the energy transfer between the system and field. Hence it is natural to check simultaneously the variation of the

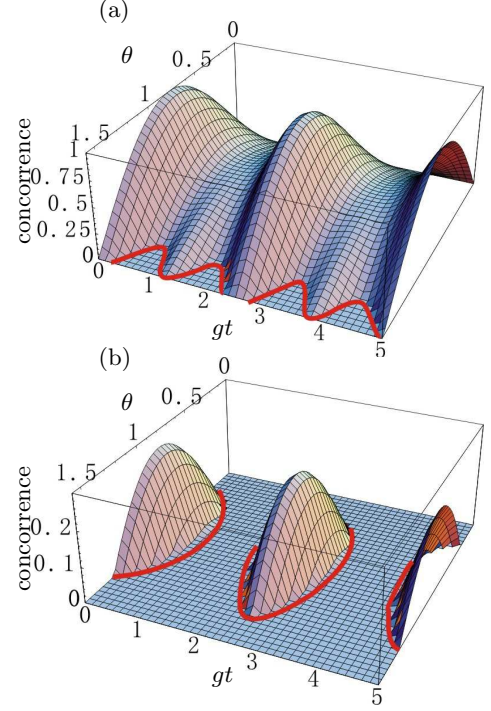


FIG. 1: The concurrence for $\varrho(t)$ versus the coupling gt and the mixing θ . (a) corresponds to $r = 1$ and (b) for $r = 0.5$. In both figures the points with vanishing concurrence have been highlighted by red lines.

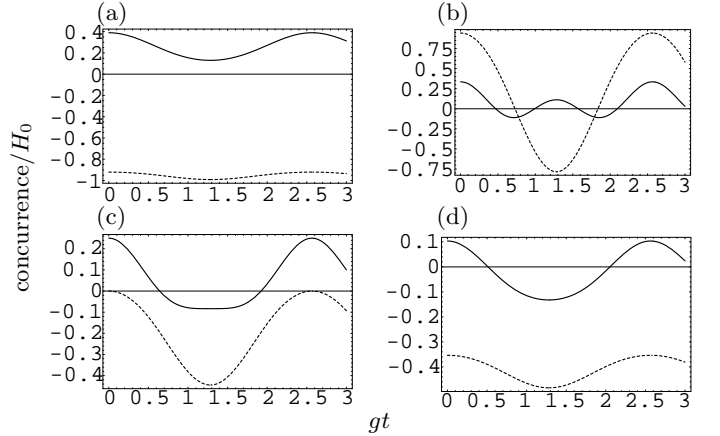


FIG. 2: The concurrence (solid line) and H_0 (dashed line) versus the rescaled time gt . For this plot we have chosen (a) $r=1, \theta=0.2$; (b) $r=1, \theta=1.4$; (c) $r=0.5, \theta=\pi/4$; (d) $r=0.5, \theta=\pi/8$. One should note that we also plot the concurrence below 0 and chose $\omega_0 = 1$ for plotting H_0 .

energy $\langle H_0 \rangle_\varrho = \langle \frac{\omega_0}{2} (\sigma_A^z + \sigma_B^z) \rangle_\varrho = \varrho_{11} - \varrho_{22}$ and concurrence in the system. The analytical relation between c and $\langle H_0 \rangle_\varrho$ is complicated and it is convenient to plot for displaying their relation. One could find the variations of $\langle H_0 \rangle_\varrho$ and concurrence is almost in-step, as shown in Fig.2. At most cases, with the increment of $\langle H_0 \rangle_\varrho$, the concurrence increases until the same value with the initial states. An exceptional situation appears in Fig.2(b), in which the second wave peak accompanies with the oppo-

site transfer of the energy to the other situations. However we could suggest that the dynamics of entanglement and its revival in this model stem from the energy transfer between the system and environment. Moreover we also note that the extreme points of the concurrence and those of $\langle H_0 \rangle_\rho$ are one-to-one. These phenomena show the intimate relation between the concurrence and the energy $\langle H_0 \rangle_\rho$ in this model.

However the relation between ESD and the transfer of energy is not direct, as shown in Fig.2. Originally ESD comes from the cutoff in the definition of concurrence. Although the physical explanation for the dynamics of concurrence can be available, ESD cannot be explained properly based on the transfer of energy in this model. A further discussion is needed beyond this model.

B. Dephasing Model

In the previous subsection we discuss the dynamics of entanglement and ESD in a dissipative two spin-1/2 system. A novel character is the direct relation between the evolution of the concurrence and energy transfer between the system and field. Another important situation for the open systems is the dephasing case, in which there is no energy transfer between the system and environment. A typical Hamiltonian for this case can be written as

$$\begin{aligned} H &= H_S + H_E + H_I, \\ H_S &= \frac{\omega_0}{2}(\sigma_A^z + \sigma_B^z) + \Omega(\sigma_A^\dagger \sigma_B^- + \sigma_B^\dagger \sigma_A^-), \\ H_E &= \sum_j \omega_j b_j^\dagger b_j, \\ H_I &= (\sigma_A^\dagger \sigma_B^- + \sigma_B^\dagger \sigma_A^-) \sum_j \Gamma_j (b_j^\dagger + b_j), \end{aligned} \quad (3)$$

where two interacting qubits is coupling with a quantized environment by H_I . The evolution operator can be given by,

$$\begin{aligned} U(t) &= \exp(-iHt) \\ &= e^{-it(\omega_0 + \sum_j b_j^\dagger b_j)} |ee\rangle\langle ee| + e^{-it(-\omega_0 + \sum_j b_j^\dagger b_j)} |gg\rangle\langle gg| \\ &\quad + e^{-it(\Omega - \sum_j \Gamma_j^2/\omega_j + \sum_j \omega_j a_{j+}^\dagger a_{j+})} |+\rangle\langle +| \\ &\quad + e^{-it(\Omega + \sum_j \Gamma_j^2/\omega_j - \sum_j \omega_j a_{j-}^\dagger a_{j-})} |-\rangle\langle -|, \end{aligned} \quad (4)$$

in which $|g(e)\rangle$ is the eigenstate of spin-1/2 particle and $|\pm\rangle = (|eg\rangle \pm |ge\rangle)/\sqrt{2}$ and $a_{j\pm} = b_j \pm \Gamma_j/\omega_j$. Choose the initial state $\rho(0) = (\frac{1-r}{4} + r|\phi\rangle\langle\phi|) \otimes |0\rangle\langle 0|$, in which $|\phi\rangle = \sin\theta|eg\rangle + \cos\theta|ge\rangle$, and at time t the reduced density matrix $\varrho(t) = \text{Tr}_E[U(t)\rho(0)U^\dagger(t)]$ can be written in the basis $\{|1\rangle = |ee\rangle, |2\rangle = |gg\rangle, |3\rangle = |+\rangle, |4\rangle = |-\rangle\}$

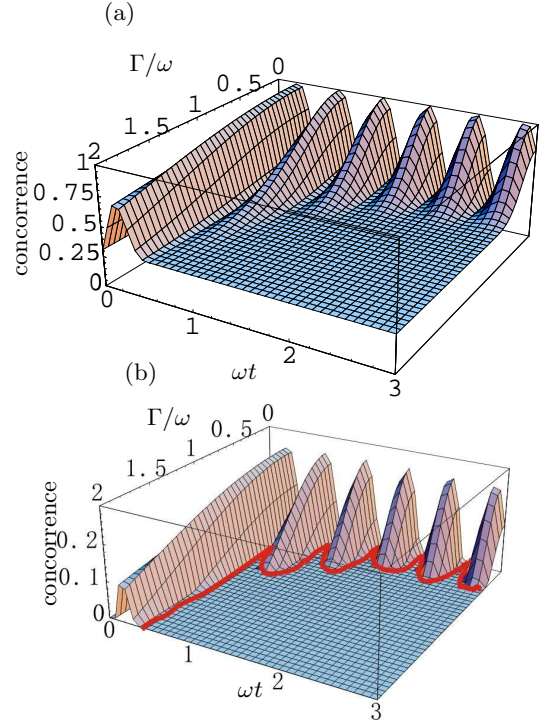


FIG. 3: The concurrence for $\varrho(t)$ versus the rescaled time ωt and the coupling Γ/ω . (a) corresponds to $r = 1$ and (b) for $r = 0.5$. In both figures we choose $\theta = \pi/20$ and $\Omega/\omega = 3$. In (b) the red line high-lightens the points of vanishing concurrence.

as

$$\begin{aligned} \varrho_{11} &= \varrho_{22} = \frac{1-r}{4}, \\ \varrho_{33} &= \varrho_{44} + r \sin 2\theta = \frac{1-r}{4} + \frac{r}{2}(1 + \sin 2\theta), \\ \varrho_{34} &= \varrho_{43}^* = -e^{-2i\Omega t} \prod_j \exp(4\Gamma_j^2 \frac{\cos \omega_j t - 1}{\omega_j^2}) \frac{r \cos 2\theta}{2} \end{aligned} \quad (5)$$

Obviously the off-diagonal terms is damping because of the decoherence factor $\prod_j \exp[4\Gamma_j^2(\cos \omega_j t - 1)/\omega_j^2]$ [12].

The concurrence for $\varrho(t)$ can be given exactly, as shown in Fig.3 with a single mode for simplicity. One notes that the entanglement does *not* decrease, but have a damping fluctuation until the same entanglement contained in the initial state, as shown in Fig. (3)(a). Another facet is that when the concurrence is zero for the initial state, it is possible for system to generate entanglement, the entanglement is also a damping-oscillating function of time, as shown in Fig.3(b). The ESD can happen at some special times. However, for the entangled initial states, there is no ESD at any time [13].

III. ENTANGLEMENT IN CLOSED SYSTEMS

In the previous section, we discuss two models coupled with a field or environment. In both models the dynamics of entanglement is dominated by the coupling with

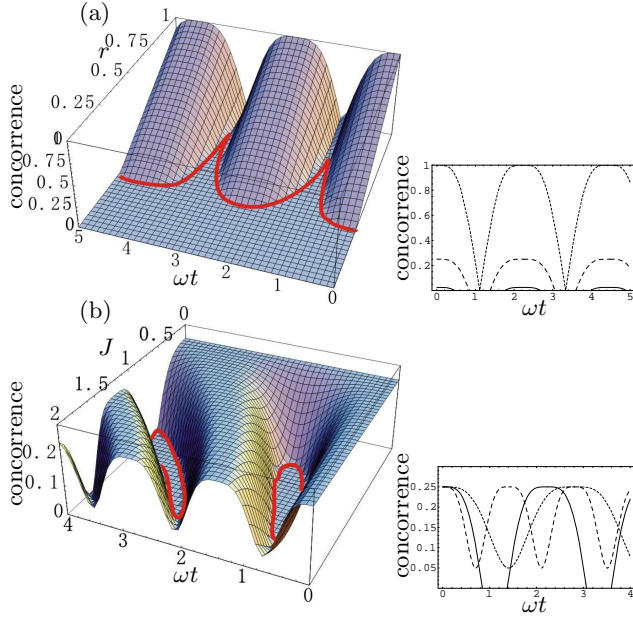


FIG. 4: The concurrence versus the time ωt and (a) the purity r , (b) the rescaled coupling $J = g/2\omega$. We have chosen $\theta = \pi/4$ and (a) $J = 1$, (b) $r = 1/2$, and the points with vanishing concurrence have been highlighting with red color. On the right of the figure (a) and (b), the sectional drawings have also provided with the same values of parameters respectively. The chosen parameters for the solid, dotted and dashed lines are (a) $r = 0.35, 1, 0.5$ and (b) $J = 1, 0.5, 2$.

the field or environment. Next we will show that ESD can also appear in closed quantum systems, in which the evolution of the system is insulated to its surroundings. Let consider two spin-1/2 particles with Ising-type interaction. The Hamiltonian is

$$H = \frac{\omega}{2}(\sigma_A^z + \sigma_B^z) + \frac{g}{2}\sigma_A^x\sigma_B^x. \quad (6)$$

The time evolution $U(t) = \exp(-iHt)$ is written in the basis $\{|1\rangle = |ee\rangle, |2\rangle = |gg\rangle, |3\rangle = |eg\rangle, |4\rangle = |ge\rangle\}$,

$$\begin{aligned} U_{11} &= U_{22}^* = \cos \frac{\lambda t}{2} - i \frac{2\omega}{\lambda} \sin \frac{\lambda t}{2}, \\ U_{12} &= U_{21} = -i \frac{g}{\lambda} \sin \frac{\lambda t}{2}, \\ U_{33} &= U_{44} = \cos \frac{gt}{2}, \\ U_{34} &= U_{43} = -i \sin \frac{gt}{2}, \end{aligned} \quad (7)$$

where $\lambda = \sqrt{4\omega^2 + g^2}$.

It is convenient to choose the initial state $\rho = \frac{1-r}{4} + r|\varphi\rangle\langle\varphi|$ with $|\varphi\rangle = \sin\theta|ee\rangle + \cos\theta|gg\rangle$. The entanglement for $\rho(t) = U(t)\rho U^\dagger(t)$ can be given easily, and the concurrence have been plotted in Fig.4. It is clear that ESD happens for some special cases and then the entanglement revives at a later time. We also choose the initial state $\rho = \frac{1-r}{4} + r|\phi\rangle\langle\phi|$ with $|\phi\rangle = \sin\theta|eg\rangle + \cos\theta|ge\rangle$. However there is no ESD. These phenomenon show again that ESD is sensitive to the initial conditions.

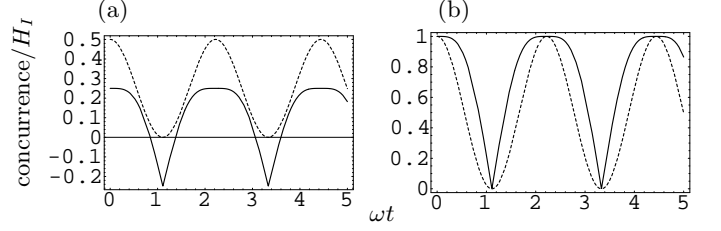


FIG. 5: The concurrence (solid line) and H_I (dashed line) versus the time ωt . For this plot we have chosen $\theta = \pi/4$ and $J = 1$. The figure (a) corresponds $r = 0.5$ and (b) for $r = 1$.

It is of great interest to check the relation between the concurrence and the energy transfer described by $\langle H_I \rangle_\rho = \langle g\sigma_A^x\sigma_B^x \rangle_\rho$. It is easily to note that the concurrence c of $\rho(t)$ is $c \propto \langle H_I \rangle_\rho / J$ and a figure has been drawn for showing their relation, displayed in Fig.5. It is very interesting to note that the evolution of concurrence and $\langle H_I \rangle_\rho$ is in-step and there is a one-to-one relation of the extreme points between the concurrence and $\langle H_I \rangle_\rho$. This phenomenon is similar to the case discussed in Sec.IIA (see Fig.2) and show that the dynamics of entanglement can be related directly to the energy transfer in this model.

IV. DISCUSSIONS AND CONCLUSIONS

Some conclusions and further discussions should be presented in this section. In this paper we discussed the dynamics of entanglement and ESD in bipartite systems. We found two very different types of the dynamics of entanglement. The first case is that there is the effect of the dissipation and vice versa, just as shown in Eq. (1) and (6). Our calculation shows the intimate relation between the concurrence and the energy transfer described by H_0 in Eq. (1) and H_I in Eq. (6); the extreme points of concurrence and the energy of H_0 or H_I under the time evolution are one-to-one. In the case without ESD, the minimums of the energy corresponds to that of the concurrence, as shown in Fig. (2)(a) and Fig. (5)(b). However, when ESD happens, the correspondence is destroyed.

In fact mathematically ESD stems from the cutoff in the definition of concurrence, and so it is difficult to find a physical interpretation. We suggest that the concurrence could be expressed as $c = \max\{0, f(E)\}$, in which $f(E)$ is a function of the energy transfer, e.g. $f(E) \propto \langle H_I \rangle_\rho / J$ in Sec. III. Then we could define a critical energy E_c , below which the system must be non-entangled, and above which the system must be entangled. When E_c is the minimal of the energy of the system, then we immediately conclude there is no ESD. Contrarily ESD could happen. Unfortunately the critical energy may be complicated and be dependent on the state as shown in Fig.2 and 5 and prevent us from giving a further analysis.

Another interesting case is the dephasing model, which has been discussed extensively in the past as one of the

standard decoherence models [14]. The evolution of concurrence in the model Eq. (3) is damping-oscillating with time until the same entanglement contained in the initial state (see Fig. 3). Moreover the oscillation of concurrence seems not directly related to the energy transfer since there is no energy transfer between system and environment. From this phenomenon it may be frustrating that the connection of concurrence and the energy is not universal. What are the reasons of the dynamics of concurrence in the dephasing case is still an open question. The phenomenon of ESD does not appear in the dephasing model when the initial state is entangled. It seems to imply that ESD intrinsically originates from the energy transfer. However the correct understanding depends on the physical meanings of the concurrence.

In conclusion we have discussed the dynamics of the entanglement and ESD in some special models. Some interesting phenomena have been presented with enlightening discussions. In our own points the main obstacle of explaining these phenomena is that it is unclear what the physical meanings of the concurrence are. Recently a great deal of works are devoted to the understanding from the energy of the system [15]. However, as shown in our paper, it may fail when there is ESD happening. More recently a physical interpretation of concurrence for the bipartite systems has been provided based on the Casimir operator in [16]. It may open another door to understand concurrence as a physical quantity.

This work was supported by NSF of China under grants 10305002 and 60578014.

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